- Characterization for Degree sequences

  σ $d_1 \geq d_2 \cdots \geq d_n \geq 0$

- Havel-Hakimi Thm

  σ is graphic $\iff$ $d_2 - 1 \cdots d_{d_1+1} - 1 \; d_{d_1+2} \cdots d_n \geq 0$ is graphic.

- Wang-Kleitman Thm

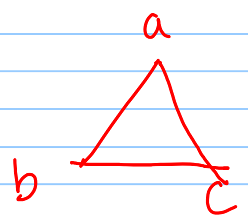
  σ is graphic $\iff$ The sequence $\underline{d_1 - 1, \cdots d_k - 1} \cdots d_{k-1} \; d_{k+1} \cdots d_n$ is graphic. where $\underline{d_k > 0}$

Recursive Algorithm to check if a given sequence is graphic

Day 2

Counting Number of spanning trees in an undirected graph

- Labeled graph



— 3 labelled spanning trees

Kirchoff's Theorem: G is an undir graph

$D - A \longrightarrow$ Laplacian. $\boxed{L_n = D - A}$

$$\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} - \begin{bmatrix} 0 & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & 0 \end{bmatrix}$$

$$\boxed{M_{11}}$$

$$\begin{array}{c} \\ 1 \\ \\ n \end{array} \begin{array}{c} 1 \\ \begin{bmatrix} \cancel{\quad} & \cancel{\qquad\qquad} \\ \cancel{\quad} & M_{11} \\ \cancel{\quad} & \end{bmatrix} \end{array}$$

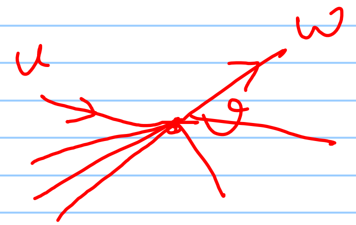
$$\boxed{\det(M_{11}) = \# \text{ of sp trees in } G}$$

Proof

Incidence matrix

$$E = V \updownarrow \begin{array}{c|ccccc|} & 1 & & & & m \\ \hline 1 & & 0 & & +1 & \\ & & 0 & & 0 & \\ & & +1 & & & \\ & & -1 & & -1 & \\ & & 0 & & & \\ n & & 0 & & 0 & \\ \hline \end{array}$$

$$\longleftarrow E \longrightarrow$$



$$E E^T = \underline{L}$$

$n \times n$

$\underline{E_1}$ obtained from E by removing 1^{st} row.

$$\boxed{E_1 E_1^T = M_{11}}$$

$$\underline{\underline{\det(E_1 E_1^T) = \det(M_{11})}}$$

$\det(E_1 E_1^T) =$ 1

$\det(A\,B)$ Cauchy-Binet Theorem

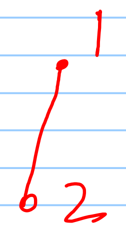
$$\underset{n}{\overset{m}{A}} \quad \underset{m}{\overset{n}{B}}$$

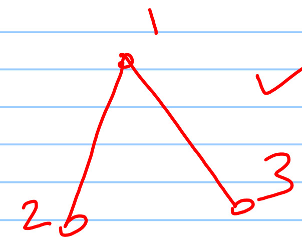
$$= \sum_{S \text{ in } \binom{m}{n}} \det(A_S) \cdot \det(B_S)$$

Applying Cauchy-Binet 1 $= \boxed{\sum_{\substack{S \text{ of} \\ n-1 \text{ edges} \\ \text{in } G}} \det(E_{1S}) \cdot \det(E_{1S}^T)}$

E_1 is a $n-1 \times m$ matrix

$$= \sum_{\substack{S \text{ of} \\ n-1 \text{ edges} \\ \text{in } G}} \underline{\det\left(E_{1S}\right)}^{2}$$





S is a set of $n-1$ edges in

a n vertex graph

$$\begin{array}{c|c} 1 & \\ 2 & \end{array} \begin{bmatrix} \cancel{+1} \\ -1 \end{bmatrix} \leftarrow E_{1S}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \cancel{+1} & \cancel{+1} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

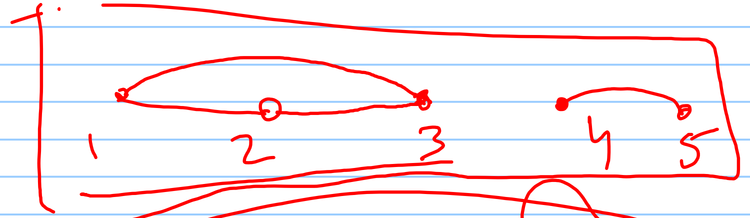
$\mathrm{Det}\left(E_{1S}\right)$

Belief: $+1$ or -1 if S forms a spanning tree in G

0 otherwise

$A_{2\underline{3}}\,A_{45}$

$A_{3\underline{4}}\,A_{4\underline{5}}\,A_{5\underline{2}}$



$$\det A = \sum_{\substack{\sigma \\ \text{perm. of} \\ \{1..n\}}} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} A_{i\sigma(i)}$$

$$\begin{array}{c|cccc}
1 & +1 & 0 & +1 & 0 \\
\hline
2 & -1 & +1 & 0 & 0 \\
3 & 0 & -1 & -1 & 0 \\
4 & 0 & 0 & 0 & +1 \\
5 & 0 & 0 & 0 & -1
\end{array} \Leftarrow E_{15}$$

Product on n-terms of the matrix one from each row

& one from each column

- If S corresps to set of $n-1$ edges that is a disconnected graph, then in the determinant of E_{1S} any product $\prod_{i=2}^{n} (E_{1S})_{i\sigma(i)} = 0$ ✓

- If it is connected, then the $\det(E_{1S}) = +1$ or -1

$$\boxed{\#\text{ of spanning trees} = \sum_{\text{spanning trees } S} \det(E_{1S})^2 = \det(M_{11})}$$

Algorithm for computing determinant of an $n \times n$ matrix

Straightforward computation of determinant

$$\sum_{i=1}^{n} (-1)^i \det(\text{minor}) \leftarrow \text{\# of arithmetic operations}$$

$$T(n) = n \cdot T(n-1)$$
$$T(1) = 1$$

$n^n \approx n!$ arithmetic operation

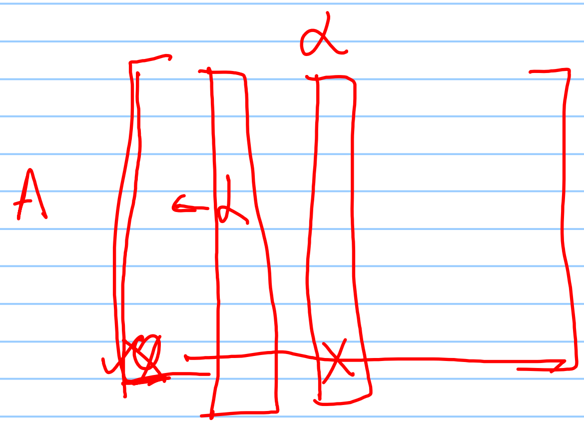
$L : D - A$

M_{11} , E , E_1

A Better Determinant Algorithm

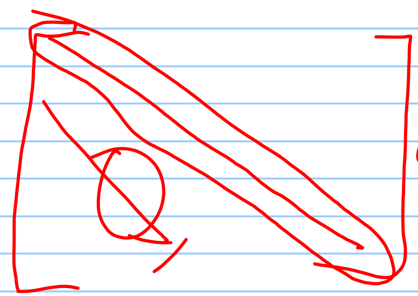
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha a_2 & a_2 & a_3 \\ b_1 + \alpha b_2 & b_2 & b_3 \\ c_1 + \alpha c_2 & c_2 & d_3 \end{vmatrix}$$

$$\frac{3n^3}{2} + n$$



To eliminate 1 element:

1) $n-1$ comparisons
2) n multiplications by a scalar
3) n additions $\longrightarrow$ DONE



$\leq 2n^3$ arithmetic ops

$$\frac{3n}{2} \times n^2$$

Analysis of the Recursive recognition of the degree sequence

$$f(d_1 \ldots d_n) = f(d_2 - 1, \ldots d_{d_1+1} - 1, \ldots d_n), \quad \text{if } d_2 - 1, \ldots d_n \geq 0$$

$$= 0, \quad \text{otherwise}$$

$$= 1, \quad \text{empty sequence}$$

RecHH(n, D)
{ If $n = 0$ return Yes
If any entry in D is -ve return No
return RecHH($n-1$, $d_2 - 1$, .. $d_{d_1+1} - 1$, ... d_n)
}

$$T(n) = 1 + n + d_1 + T(n-1) + \text{sorting time}$$

Running time is analyzed by the following recurrence

$$T(n) = 2n\log n + T(n-1)$$

$$T(0) = 1$$

$$T(n) \leq \sum_{i=0}^{n} 2n\log n = \underline{2n^2\log n}$$

- Algorithmic Problems – Graph Theoretic or Geometric in nature

Input – graph or a set of line segments
or ... ..

Input has a size – number parameters represented

n, m – vertices & edges

n^2 input values to describe the graph.

Identify the solution with a proof of correctness

Analyze the # of operations in the algorithmic solution

Formally defined the running time function $T(n)$ for the algorithm.